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MODEL OF WATER INFLUX TO A PERFECT WELL WITH ALLOWANCE FOR THE WATER LOSS BY THE OVERLYING CLAY LAYER

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Consideration is given to a mathematical model of water influx to a perfect well drilling in a homogeneous water-bearing stratum. The novelty of the problem is in the method of allowing for the additional water influx from the overlying swelling clay layer to the stratum. The model of water loss by a swelling clay layer, proposed by the authors earlier, is used for calculation of the water influx. This model is based on the generalization of filtration-consolidation theory to the case where the mass of the solid phase of a porous skeleton changes due to the fluid crossflow during the processes of swelling and shrinkage under the action of osmotic pressure.

Introduction. The model of rheological properties of porous media with a swelling skeleton, which is important for understanding the physicomechanical behavior of swelling media in general, was proposed by the present authors earlier [1]. Swelling systems in nature are soils, clay rocks, certain polymers, and polymolecular systems of living organisms. The application of a theory enabling one to use both the mechanical and physicochemical parameters of the system is crucial in describing the properties of such systems. Generalization of filtration-consolidation theory to the case of porous media with a skeleton of variable mass provides an example of such a theory, as we believe. The mass of the porous skeleton changes due to swelling. Hence the main objectives of the present investigation are:

(1) development of a physicomathematical model of the process of swelling that enables one to describe the processes of mass transfer in deformable systems in a unified context;

(2) computational experiment carried out using a well-studied (in terms of the presence of experimental data on the process) system for comparing experimental and calculation results as an example.

The first objective has been accomplished in [1]. The water loss by a clay layer under the action of constant load was considered as an example. Curves for the shrinkage and deformation rate of this layer were constructed. An advantage of the model developed is that no assumptions of the rheological behavior of the layer are made in advance. The constant of the rate of absorption of water by clay in swelling plays the role of the rheological parameter. Nonetheless, the problem on the actual natural object for which the relations obtained would play a crucial role remains to be solved. As such an object we will consider a natural water-bearing (aqueous) stratum drilled-in by a symmetric perfect well and overlapped by a powerful mass of swelling clay rocks. The selection of such an object is not random. It is well known, e.g., that in long-term exploitation of sweet-groundwater pools, i.e., at times close to the exhaustion of guaranteed water supplies, allowance for the additional water influx from the overlying clay mass to the stratum becomes much needed. Prediction of such an influx would enable one to increase the service life of a water intake. The problem has become pressing in recent times in connection with the contamination of natural groundwater, which leads to a substantial reduction in the service lives of acting water intakes. The solution of this problem is the second objective of the present investigation.

Model of Water Influx to a Perfect Well with Allowance for the Water Loss by the Overlying Clay Layer. The equation of filtration of a compressible fluid in a compressible porous medium [2] has the form

$$\partial p/\partial t = \chi_a \Delta p \,. \qquad (1)$$

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According to [3], filtration in a low-permeability clay layer is vertical, whereas in a water-beating stratum, it is horizontal. Consequently, in a cylindrical coordinate system, for the influx to a symmetric perfect well we have

$$\frac{\partial p}{\partial t} = \chi_a \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + \chi_a \frac{\partial^2 p}{\partial z^2} . \tag{2}$$

Averaging Eq. (2) over the stratum height (over the coordinate z from 0 to H), for p we obtain

$$\frac{\partial p}{\partial t} = \chi_a \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + J , \quad J = \frac{\chi_c}{H} \frac{\partial w}{\partial z} (z = 0) . \tag{3}$$

The boundary conditions on the well and the external boundary respectively have the form

$$p (r_b) = p_b , \quad p (r_\infty) = p_\infty . \tag{4}$$

For the initial pressure distribution in the water-bearing layer we use the dependence

$$p (r, 0) = (p_\infty - p_b) [1 - \exp (- (r - r_b)/r_b)] + p_b . \tag{5}$$

We have obtained the model of water influx to a symmetric perfect well in the water-bearing stratum with allowance for the water influx from the overlying clay layer. In calculating the distribution of the pressure w in the clay layer, we must take into account that the lower pressure in the layer takes on the value of the running pressure in the water-bearing stratum (at each r point), i.e., we have a two-dimensional problem of nonstationary filtration of two neighboring strata (clay and water-bearing ones). Next, after the procedure of making all equations dimensionless, the mathematical model obtained is solved numerically.

Dimensionless Problem in the Clay Layer. Let us introduce dimensionless variables $\bar{\sigma} = \sigma^f/\Gamma$, $\bar{w} = w/\Gamma$, $\bar{\pi} = \pi/\Gamma$, $\bar{\tau} = t/T_0$, $\bar{q} = q/U$, $\bar{z} = z/z_0$, and $T_0 U/z_0 \approx 1$. For evaluation of the characteristic velocity U we have $U = K\Gamma/(\rho_w g z_0)$, $\rho_w g k/\eta = K$. Taking $\Gamma \approx 2$ MPa and $z_0 \approx 1$ m, we may roughly evaluate U. The quantity K is the physical characteristic of the stratum. According to experimental data for clays, the value of the filtration coefficient is $\approx 10^{-3}$ m/day. After transformations, we write the resulting system of equations in which all unknown variables will represent dimensionless quantities:

$$\bar{\sigma} + \bar{w} = 1 , \tag{6}$$

$$\partial \vartheta / \partial \bar{\tau} + \partial \bar{q} / \partial \bar{z} = 0 , \tag{7}$$

$$\bar{q} = - \partial \bar{w} / \partial \bar{z} , \tag{8}$$

$$f = V_s (\delta_1 \exp \vartheta - 1) , \quad \delta_1 = V_0^{(0)} / v_s , \quad V_0^{(0)} = 1 , \quad V_s = 2/3 , \tag{9}$$

$$\partial f / \partial \tau = \bar{\alpha} (\bar{\pi} - \bar{\sigma}) , \quad \bar{\alpha} = T_0 \alpha \Gamma , \tag{10}$$

$$\bar{\pi} = \delta_3 / (\delta_1 \exp \vartheta - 1) , \quad \delta_3 \approx 2 . \tag{11}$$

The initial conditions are as follows:

$$\bar{w} (\bar{z}, 0) = (\bar{p} - 1) \exp (- \bar{\alpha} \bar{z}) + 1 , \quad \bar{\alpha} = 10^{-3} . \tag{12}$$

The boundary conditions are

$$\bar{w}(1, \bar{\tau}) = 1 , \quad \bar{w}(0, \bar{\tau}) = \bar{p} . \tag{13}$$

The system of equations (6)–(13) is solved by the marching method. A difference scheme for this system is presented below.

Dimensionless Problem in the Aqueous Stratum. Let us write Eq. (3) in dimensionless form. For this purpose we introduce new variables $\bar{p} = p/\Gamma$, $\bar{r} = r/r_\infty$, and $\bar{J} = J/\Gamma$, where r_∞ is of the order of 100 m. Then we have

$$\frac{\partial \bar{p}}{\partial \bar{\tau}} = \frac{\chi_a z_0}{U r_\infty^2} \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{p}}{\partial \bar{r}} \right) + \frac{z_0 \bar{J}}{U} , \quad \bar{J} = \frac{k}{\beta_a \eta H z_0} \frac{\partial \bar{w}}{\partial \bar{z}} = \frac{k \rho_w g \,(\partial \bar{w}/\partial \bar{z})}{\eta \beta_a \rho_w g H z_0} = \frac{K}{\beta_a \rho_w g H z_0} \frac{\partial \bar{w}}{\partial \bar{z}} . \tag{14}$$

Carrying out simple mathematical transformations, we obtain the dimensionless equation for pressure in the water-bearing stratum

$$\frac{\partial \bar{p}}{\partial \bar{z}} = A \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{p}}{\partial \bar{r}} \right) + B \frac{\partial \bar{w}}{\partial \bar{z}} . \tag{15}$$

We evaluate each of the combinations obtained: $A = \dfrac{\chi_a z_0}{U r_\infty^2}$, $B = \dfrac{K}{U \beta_a g \rho_w H}$, and $U = K \dfrac{\Gamma}{\rho_w g z_0} \Rightarrow B = \dfrac{z_0}{\beta_a \Gamma H}$. Since we have $A = 5 \cdot 10^2$, Eq. (15) takes the form

$$\bar{A} \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{p}}{\partial \bar{r}} \right) + \frac{\partial \bar{w}}{\partial \bar{z}} = 0 , \quad \bar{A} = \frac{A}{B} . \tag{16}$$

We note that the last equation has been written with allowance for the smallness of the factor preceding $\partial \bar{p}/\partial \tau$. The initial conditions are as follows:

$$\bar{p}(\bar{r}, 0) = \delta_4 (1 - \exp[-(\bar{r} - \varepsilon)/\varepsilon]) , \quad \varepsilon = r_b/r_\infty , \quad \delta_4 \approx p_\infty/\Gamma . \tag{17}$$

The boundary conditions of the first kind are

$$\bar{p}(\varepsilon, \bar{\tau}) = 0 , \quad \bar{p} = 1 , \tag{18}$$

those of the second kind are

$$\bar{p}(\varepsilon, \bar{\tau}) = 0 , \quad \partial \bar{p}/\partial \bar{r}\, (\bar{r} = 1) = \text{const} . \tag{19}$$

Difference Scheme of the Problem. To solve (6)–(19) we select an explicit difference scheme that turns out to be quite stable to changes in the model's parameters. We write the difference scheme for the problem in the clay layer. It is clear that we may reduce the number of unknowns in the system thus diminishing the number of equations to make calculation more convenient. We are able to attain this by simple substitution

$$\partial \vartheta/\partial \bar{\tau} + \partial^2 \bar{w}/\partial \bar{z}^2 = 0 , \quad \vartheta = \ln \left[((f + 1) V_s^{-1} + 1)/\delta_1 \right], \quad \partial f/\partial t = \bar{\alpha} \left[\delta_3 (\delta_1 \exp \vartheta - 1)^{-1} - 1 + \bar{w} \right].$$

Thus, the number of unknowns in our system has been reduced to three: $\bar{w}$, ϑ, and f. We write the difference equation for the function f:

$$\frac{f[j, i] - \bar{f}[j, i]}{h_\tau} = \bar{\alpha} \left[\frac{\delta_3}{\delta_1 \exp \vartheta - 1} - 1 + \bar{w}[j, i] \right],$$

where $\bar{f}[j, i]$ is the value of the function f on the previous time step. Now we pass to the difference equation for shrinkage:

$$\vartheta\,[j, i] = \ln\left[(f\,[j, i] + 1)\; V_s^{-1} + 1\right]/\delta_1\,.$$

As is seen from the difference equations given above, the value of the function f and the shrinkage are found directly, whereas to determine the pressure in the clay layer we have to resort to the so-called marching method. First we write the difference analog of Eq. (7):

$$\frac{\vartheta\,[j, i] - \hat{\vartheta}\,[j, i]}{h_\tau} + \frac{\bar{w}\,[j, i-1] - 2\bar{w}\,[j, i-1] + \bar{w}\,[j, i+1]}{h_z^2} = 0\,.$$

Here $\hat{\vartheta}[j, i]$ is the value of the shrinkage function on the previous time layer. It is noteworthy that, in all the difference equations given above, the index j corresponds to a step in the r direction, whereas the index i corresponds to a step in the z direction. The values of the time step h_τ and the z step h_z will be discussed below. We write the marching formula:

$$A_i\, y_{i-1} - C_i\, y_i + B_i\, y_{i+1} = F_i\,.$$

It is clear that in our case we have

$$A_i = 1\,;\quad B_i = 1\,;\quad C_i = 2\,;\quad F_i = \left[(\vartheta\,[j, i] - \vartheta\,[j, i])\; h_z^2\right]/h_\tau\,.$$

Next we use the standard variant of the marching method, i.e., construction of the marching coefficients

$$\alpha_{i+1} = B_i/(C_i - \alpha_i A_i)\,,\quad \beta_{i+1} = (A_i \beta_i + F_i)/(C_i - \alpha_i A_i)\,.$$

Accordingly the first marching coefficients have been computed in terms of the boundary conditions. Finally, we compute the values of pressure in the clay layer in reverse marching.

Now we pass to description of the difference equations for the water-bearing stratum. Let us simplify the basic equation for the clay stratum by replacing the variables: $x = \ln(r) \Leftrightarrow r = \exp(x)$. Then we obtain

$$\bar{A}\exp(-2x)\,\frac{\partial^2 \bar{p}}{\partial \bar{x}^2} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0\,.$$

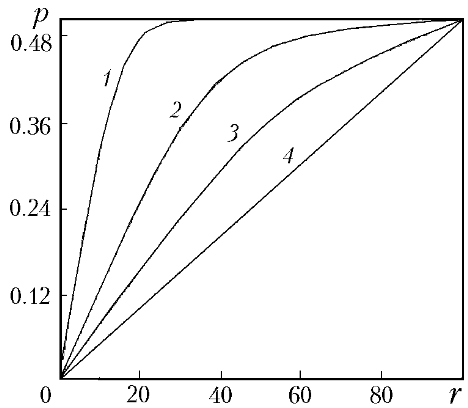
It is clear that this equation is easily solvable numerically using the marching method. We write the difference analog of Eq. (16):

$$\bar{A}\exp(-2jh_x)\,\frac{\bar{p}\,[j-1] - 2\bar{p}\,[j] + \bar{p}\,[j+1]}{h_x^2} + \frac{w\,[j, 1] - w\,[0, j]}{h_z} = 0\,.$$

In this case, in the marching method, we have

$$A_i = 1\,,\quad B_i = 1\,,\quad C_i = 2\,,\quad F_i = \frac{(w\,[j, 1] - w\,[0, j])\; h_x^2}{h_z\, \bar{A}\exp(-2jh_x)}\,.$$

To complete consideration of the numerical scheme of the problem we must write the values of steps and the method by which the grid nodes are prescribed:



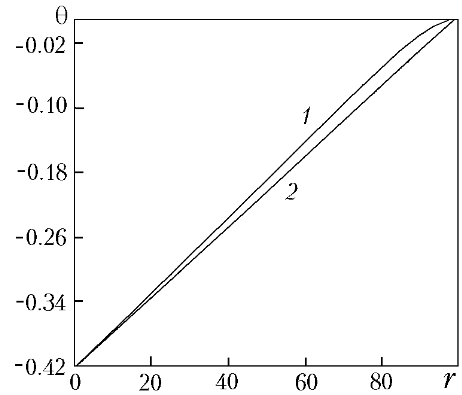


Fig. 1. Pressure distribution in the stratum at different instants of time: 1) initial distribution; 2 and 3) with allowance for the effect of swelling of the overlying layer for the intermediate instant of time and without allowance for it at the same instant of time; 4) stationary pressure distribution in the stratum.

Fig. 2. Radial distribution of the shrinkage of the clay layer with allowance for the effect of swelling of the overlying layer (1) and without allowance for it.

$$\omega = \left\{ (x_j, z_i, t_l) : x_j = jh_x, \, h_x = \frac{\ln(\varepsilon)}{N_x}; \, z_i = ih_z, \, h_z = \frac{1}{N_z}; \, t_l = lh_\tau, \, h_\tau = \frac{T}{N_t} \right\}.$$

For the explicit difference scheme, the stability condition in our case is known to be $h_\tau / h_z^2 \leq 0.5$. Accordingly the density of subdivision of the grid in the coordinates x, z and the time t, determined by the numbers N_x, N_z, and N_t, is selected from the stability condition. Numerical experiments have shown that the scheme is quite stable to the change in the density of subdivision of the grid.

The general idea of the numerical solution is as follows. The problem of finding the pressure in the water-bearing stratum is first solved on each time step, and then, at each point of subdivision of the x axis, the problem for the clay stratum with a boundary condition equal to the water-bearing-stratum pressure at this point is solved. Accordingly the water flow from the clay layer to the water-bearing stratum is found on each time step during the computation of the pressure values in the water-bearing stratum.

Analysis of the Results. Figure 1 shows the pressure distribution in the stratum at different instants of time. The step of subdivision selected along r is 10^{-2}, so that the value $r = 1$ corresponds to 100 steps of the grid. The stationary pressure distribution in the stratum is linear, which is consistent with the physical ideas of the course of the process. The difference from the case where the distinctive features of the deformation of a swelling layer are disregarded manifests itself, first, in different velocities with which the corresponding pressure values in the stratum and the clay layer are reached. Thus, if we take into account the distinctive features of the water loss by the swelling layer, the corresponding pressure values are attained over a longer period of time (curves 3 and 4). Second, the difference between the cases manifests itself in the shrinkage values (Fig. 2). The selected step of time subdivision is 10^{-3}, so that the value $t = 1$ corresponds to 1000 steps of the grid. The special properties of swelling manifest themselves in the nonlinear dependence of the average shrinkage on the radius at the stationary stage compared to the case where the effect of swelling of the clay layer is disregarded, which is also quite consistent with the physical ideas of the process.

Conclusions. Realization of the model [1] for the problem of water intake from a perfect well with allowance for the water loss by the overlying swelling clay layer has shown a substantial dependence of the course of the process on the physical characteristics of the deformable layer. Allowance for the fact that the layer possesses an additional mechanical resistance to shrinkage is manifested as pressure reaching more slowly the corresponding stationary values and as shrinkage rates lower than those obtained without allowance for the swelling factor. Consequently, we may speak of the longer service life of the stratum with sweet-groundwater supplies, which is ensured by the additional water loss by the overlying clay layer.

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NOTATION

A, B, and $\bar{A}$, dimensionless numbers; A_i, B_i, C_i, and F_i, coefficients in the marching formula; f, content of bound water in the clay; g, free fall acceleration; h, size of the grid step; H, thickness of the water-bearing stratum; i, j, and l, Nos. of vertical, radial and time step respectively; J, water influx from the clay; k, permeability of the water-bearing stratum; K, filtration coefficient, m/day; N_x, N_z, and N_t, numbers determining a grid step along the coordinates x, z and in time t respectively; p, pressure in the water-beating stratum; p_b, pressure on the well; p_∞, pressure at the external boundary; q, rate of filtration in the clay layer; r, radius; r_b, radius of the well; r_∞, radius of the external boundary; t, time; T_0, characteristic time of the process; U, characteristic rate of filtration in the stratum; V_0, representative volume; $V_0^{(0)}$, initial value of the medium's representative volume; V_s, volume of the solid phase; w, pressure in the clay layer; x, logarithmic radius; z, vertical coordinate; z_0, thickness of the clay layer; α, constant of the rate of absorption of water by clay in swelling; β_a, coefficient of compressibility of the water-bearing stratum; χ_a, piezoconductivity of the water-bearing stratum; χ_c, piezoconductivity of the clay layer; Δ, Laplace operator; Γ, external load; ε, ratio r_b/r_∞; δ_i (i = 1, 2, 3, and 4), dimensionless numbers; η, viscosity of water; ϑ, shrinkage; π, osmotic pressure; ρ_w, density of water; σ^f, effective stress; ω, method of prescription of the grid nodes. Subscripts and superscripts: 0, characteristic scale; a, water-bearing (aqueous) stratum; b, well (bore); c, clay; ∞, external boundary; f, effective; s, solid phase; w, water; x, radial step; z, vertical step; τ, time step; $\hat{}$, partial time derivative; $\bar{}$, averaged quantity.

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